(15)

1. $x' = 2 \quad x'' = 4 \quad \lambda = \frac{1}{4} \quad y = x^2$

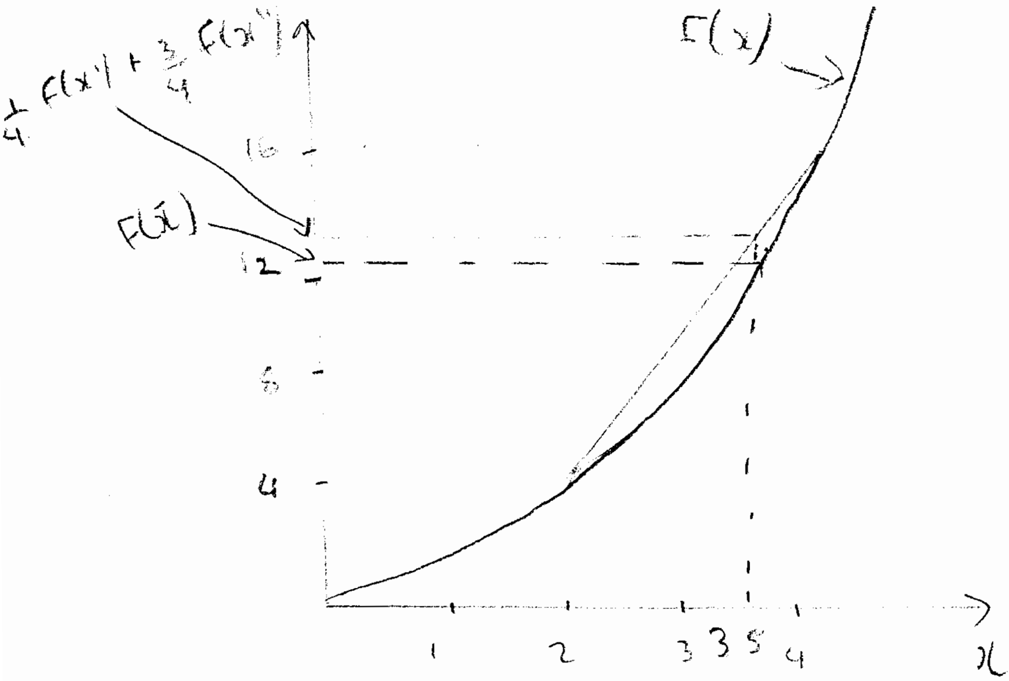
$\Rightarrow \bar{x} = \frac{1}{4} \cdot 2 + \frac{3}{4} \, 4 = 3.5$

$f(\bar{x}) = 3.5^2 = 12.25$

$f(x)$ is s. convex if $f(\bar{x}) < \lambda f(x') + (1-\lambda) f(x''$

$= \frac{1}{4} \cdot 4 + \frac{3}{4} \, 16$

$= 13$ ✓



True $\forall \; x', x''$ $\quad$ $\bar{x} = \lambda x' + (1-\lambda) x''$

LHS $\quad f(\bar{x}) = (\lambda x' + (1-\lambda) x'')^2$

$= \lambda^2 x'^2 + 2\lambda(1-\lambda) x' x'' + (1-\lambda)^2 x''^2$

$<$ RHS $= \lambda x'^2 + (1-\lambda) x''^2$

$1 - 2\lambda + \lambda^2$

$\Rightarrow$ Need $\quad (\lambda^2 - \lambda) x'^2 + ((1-\lambda)^2 - (1-\lambda)] x''^2 + 2\lambda(1-\lambda) x' x'' < 0$

$\Rightarrow (\lambda^2 - \lambda)[x'^2 + x''^2 - 2x'x''] < 0$

$\Rightarrow \lambda(\lambda - 1)\underbrace{(x' - x'')^2}_{\text{always} > 0} < 0$

$+ \quad -$ ✓

2. $y = e^{x^2}$

$$f(x) \approx f(0) + x f'(0) + \frac{x^2}{2!} f''(0)$$

$$f(0) = e^{0^2} = 1$$

$$f'(x) = 2x e^{x^2} \Rightarrow f'(0) = 0$$

$$f''(x) = 2e^{x^2} + 4x^2 e^{x^2} \Rightarrow f''(0) = 2$$

$$\Rightarrow e^{x^2} \approx 1 + x \times 0 + \frac{x^2}{2!} \times 2 = 1 + x^2$$

eg $x = 0.05 \Rightarrow e^{x^2} \approx 1 + 0.05^2 = 1.0025$

Actually $e^{0.05^2} = 1.002503$

$\therefore$ error $= 0.000003$

3. $x_1 = 20L_1^{1/2}$ $\quad L_1 + L_2 = 100$

$x_2 = L_2$

a. $mPL_1 = \frac{dx_1}{dL_1} = 10L_1^{-1/2}$

$mPL_2 = \frac{dx_2}{dL_2} = 1$

b. $L_2 = x_2 \qquad L_1 = \left(\frac{x_1}{20}\right)^2 = \frac{1}{400}x_1^2$

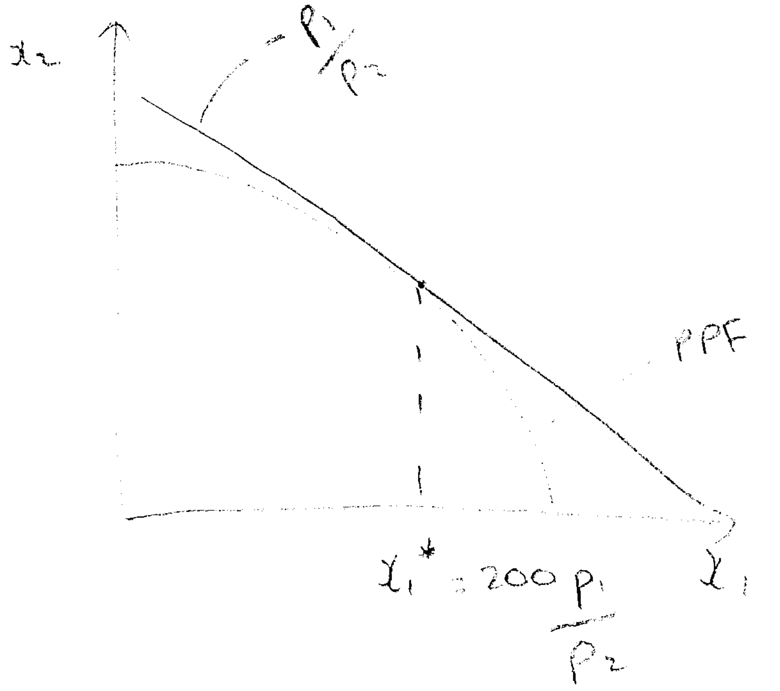
c. Use b. $\Rightarrow L_1 + L_2 = 100 = x_2 + \frac{1}{400}x_1^2$

$\Rightarrow x_2 = 100 - \frac{1}{400}x_1^2$

$\frac{dx_2}{dx_1} = -\frac{1}{200}x_1 \quad < 0 \quad \forall x_1 > 0$

$\frac{d^2x_2}{dx_1^2} = -\frac{1}{200} \quad < 0 \quad \forall x_1, x_2$

$\therefore$ PPF slopes $\downarrow$ & is strictly concave



d. $\max y = p_1 x_1 + p_2 x_2$

$$= p_1 x_1 + p_2 \left(100 - \frac{1}{400} x_1^2\right)$$

$$\frac{dy}{dx_1} = p_1 - \frac{1}{200} p_2 x_1$$

$$= 0 \quad \text{when} \quad x_1^* = 200 \frac{p_1}{p_2}$$

e. $$\frac{dx_1^*}{dp_1} = \frac{200}{p_2} > 0 \quad \text{ie when } p_1 \uparrow,\ x_1 \uparrow$$

$$x_2^* = 100 - \frac{1}{400} (x_1^*)^2 = 100 - \frac{1}{400} 40{,}000 \frac{p_1^2}{p_2^2}$$

$$= 100 - 100 \frac{p_1^2}{p_2^2}$$

$$\frac{dx_2^*}{dp_1} = -200 \frac{p_1}{p_2^2} < 0$$

ie $p_1 \uparrow$ $x_2 \downarrow$

(see illustration)

(Note: Comp Stats easy when explicit functions. Qu. 4 about doing it $\bar{c}$ general function form.)

4. $\pi = (p(x) - t]x - c(x)$

a. $\frac{d\pi}{dx} = \pi'(x) = p'(x).x + (p(x) - t) - c'(x) = 0$

$$\frac{d^2\pi}{dx^2} = p''(x).x + p'(x) + p'(x) - c''(x)$$

$$= \underbrace{p''(x).x + 2p'(x)}_{-ve} - \underset{+}{c''(x)}$$

$$< 0 \quad \forall x$$

$\therefore$ SOSC is satisfied - global max.

b. Take TD of FOC (ie use imp. function rule)

or may use 3 step method from lecture notes.

$$\Rightarrow p''(x)x.dx + p'(x)dx + p'(x)dx - dt - c''(x)dx = 0$$

$$\Rightarrow (p''(x).x + 2p'(x) - c''(x)]dx = dt$$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{(-)} < 0 \text{ from SOSC}$$

$\therefore$ As t falls, x will rise

$$\frac{dp}{dt} = \frac{dp}{dx} \cdot \frac{dx}{dt} = \frac{\overset{<0}{p'(x)}}{(-)} > 0$$

$\therefore$ As t falls, p will fall

5. $A\underset{\sim}{x} = d \Rightarrow$

$$Q + bp^c = a$$
$$Q - ep^e = c$$
$$p^c - p^e = t$$

$$\Rightarrow \begin{bmatrix} 1 & b & 0 \\ 1 & 0 & -e \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} Q \\ p^c \\ p^e \end{bmatrix} = \begin{bmatrix} a \\ c \\ t \end{bmatrix}$$

$$\Rightarrow p^c = \frac{\begin{vmatrix} 1 & a & 0 \\ 1 & c & -e \\ 0 & t & -1 \end{vmatrix}}{|A|} = \frac{1(-c+et) - a(-1)}{1(0+e) - b(-1)}$$

$$= \frac{a - c + et}{b+e}$$

6. $$\begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & -2 \\ 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$|A| = 3(4-2) - 1(0+4) + 1(0-2) = 0$$

$\therefore$ Not a unique solution

$\therefore r(A) < 3 \quad \because |A| = 0$

$r(A) = 2 \quad \because \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} \neq 0$

$$B = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 2 & -1 & 4 & 3 \end{bmatrix}$$

$\begin{vmatrix} 3 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & -1 & 3 \end{vmatrix} = 12 > 0 \quad \therefore r(B) = 3 > r(A)$

so equations are inconsistent.

8. $U = [0.3x_1^{-3} + 0.7x_2^{-3}]^{-1/3}$

$$dU = \frac{\partial U}{\partial x_1} . dx_1 + \frac{\partial U}{\partial x_2} . dx_2 = 0$$

$$\Rightarrow \frac{dx_2}{dx_1} = - \frac{\partial U/\partial x_1}{\partial U/\partial x_2}$$

$$= - \frac{\left[-\cancel{\tfrac{1}{3}} \cancel{(\cdot)^{-4/3}} . -\cancel{3} \times 0.3 x_1^{-4} \right]}{\cancel{-\tfrac{1}{3}} \cancel{(\cdot)^{-4/3}} . \cancel{-3} \times 0.7 x_2^{-4}}$$

$$= - \frac{0.3}{0.7} . \frac{x_2^4}{x_1^4} \quad \text{or} \quad 0.43 \frac{x_2^4}{x_1^4}$$

7. $U(x, p) \qquad p = g(x)$

$$\frac{dU}{dx} = \frac{\partial U}{\partial x} + \frac{\partial U}{\partial p} \frac{dP}{dx} \quad \text{or} \quad \frac{dU}{dx} = U_x + U_p . g'(x)$$

The total derivative of U w.r.t x takes into account the partial effect of a change in x, U_x (holding P constant) AND the effect that P has on utility given that a change in x will trigger a change in P ($g'(x)$) which in turn changes U (U_p). Here the total derivative will be smaller than the partial because, while an increase in x causes U to rise, this is offset by the increase in P, which causes U to fall.