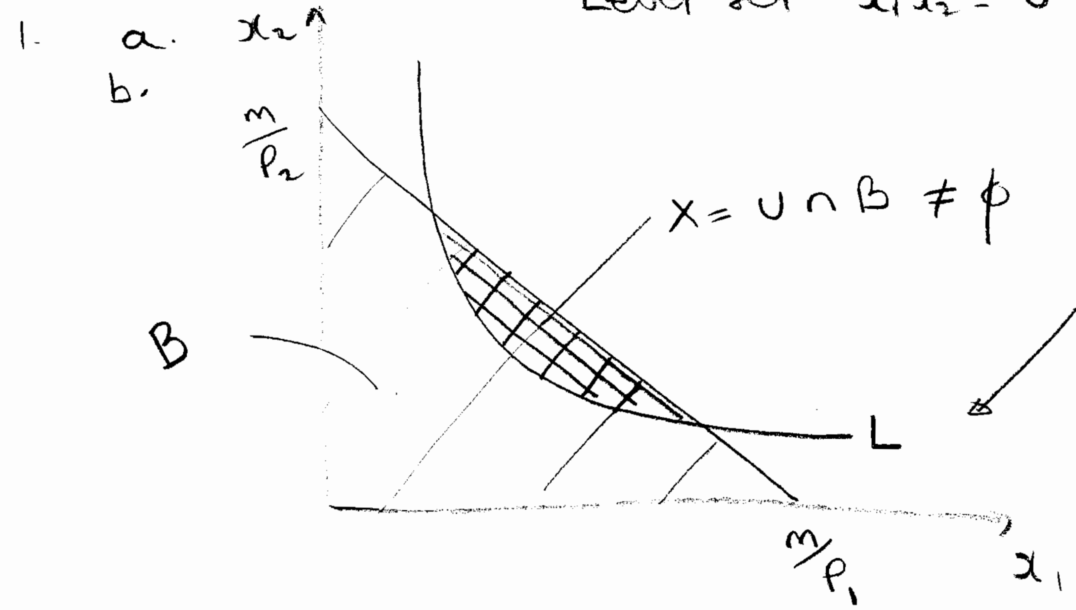
Mid-Semester Exam, May 2004

Level set $x_1 x_2 = \bar{U} \Rightarrow x_2 = \frac{\bar{U}}{x_1}$

1. a.
b.



(Note: $U = x_1 x_2 > \bar{U}$ is the set of socially acceptable utility, ie $x_1 x_2 \leq \bar{U} \Rightarrow$ poverty)

c. Is X closed? No - it doesn't contain lower boundary

open? No - it does contain upper boundary, so can't find $\varepsilon > 0$ s.t $N_\varepsilon(x) \subset X$ for points on boundary.

Bounded? Yes - can: for every $x_0 \in X$, $\exists \varepsilon < \infty$ s.t $X \subset N_\varepsilon(x_0)$ i.e. can find an epsilson $\forall x_0$ in X s.t all of X is contained with epsilon neighbourhood of x_0.

Compact? No - since it's not closed, & compact means closed and bounded.

Convex? Yes - all convex combos lie in set (including straight line boundary) ie $\forall \lambda \in (0,1)$, $\bar{x} = \lambda x' + (1-\lambda) x''$ belongs to X.

S.convex? No: straight line means some convex combos NOT in INTERIOR of X.

2. $y = e^{-x^2} \qquad x \geq 0$

a. $\lim_{x \to 0} e^{-x^2} = 1 \qquad \lim_{x \to \infty} e^{-x^2} = 0$

b. $\frac{dy}{dx} = -2x e^{-x^2} < 0 \quad \forall x \geq 0$

$$\frac{d^2y}{dx^2} = -2x \cdot -2x e^{-x^2} - 2e^{-x^2}$$

$$= (4x^2 - 2) e^{-x^2}$$

$$= 0 \text{ when } 4x^2 - 2 = 0$$

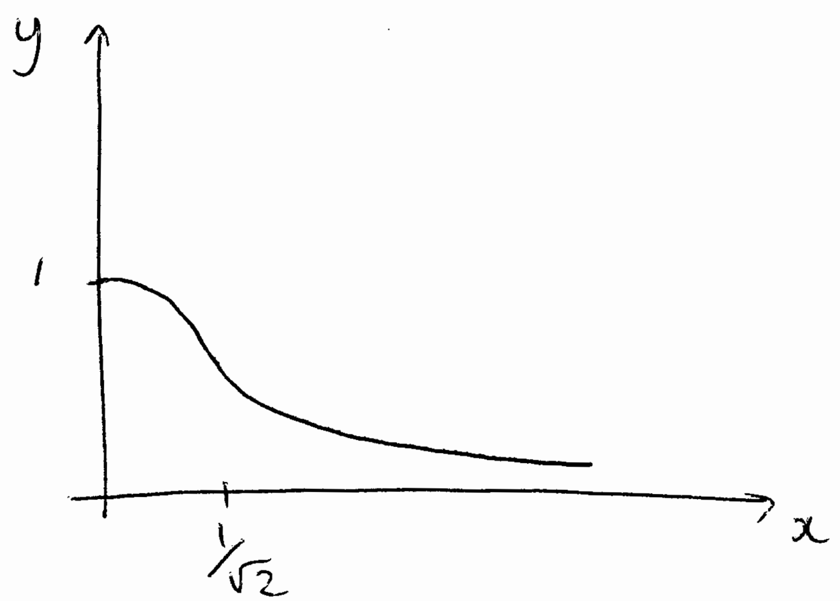
$$x^2 = \tfrac{1}{2}$$

$$x = \frac{1}{\sqrt{2}}$$

$\therefore \frac{d^2y}{dx^2} > 0$ when $x > \frac{1}{\sqrt{2}} \Rightarrow$ s. convex

< 0 " $x < \frac{1}{\sqrt{2}} \Rightarrow$ s. concave

when $x = 0$, $y = 1$.



3. a. $|P - \lambda I| = \begin{vmatrix} 2/3 - \lambda & 1/3 \\ 1/3 & 2/3 - \lambda \end{vmatrix}$

$= \left(\frac{2}{3} - \lambda\right)^2 - 1/9 \Rightarrow 4/9 - 4/3\,\lambda + \lambda^2 - \frac{1}{9}$

$= 0 \Rightarrow \lambda^2 - 4/3\,\lambda + \frac{1}{3} = 0 \Rightarrow 3\lambda^2 - 4\lambda + 1 =$

$\Rightarrow (3\lambda - 1)(\lambda - 1) = 0$

$\lambda_1 = 1/3, \quad \lambda_2 = 1$

b. $\lambda_1 = 1/3 \Rightarrow \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow q_1 = -q_2$

Normalise $\tau \quad q_1^2 + q_2^2 = 1$

$\Rightarrow 2q_2^2 = 1 \Rightarrow q_2 = \frac{1}{\sqrt{2}}, q_1 = -\frac{1}{\sqrt{2}}$

$\lambda_2 = 1 \Rightarrow \begin{pmatrix} -1/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \Rightarrow q_1 = q_2 = \frac{1}{\sqrt{2}}$

$\therefore Q = \begin{bmatrix} -1/\sqrt{2} & \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

(Check orthogonal : $Q^TQ = \begin{bmatrix} -1/\sqrt{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓)

c. $x^2 = \overbrace{Q \Lambda^2 Q^T}^{P^t} x_0$ (since P is symmetric)

$$= \begin{bmatrix} -1/\sqrt{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1/9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{9\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{9\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{18} + \frac{1}{2} & -\frac{1}{18} + \frac{1}{2} \\ -\frac{1}{18} + \frac{1}{2} & \frac{1}{18} + \frac{1}{2} \end{bmatrix} \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} 10/18 \times 6 + \frac{8}{18} \times 9 \\ 8/18 \times 6 + 10/18 \times 9 \end{bmatrix} = \begin{bmatrix} 7.\dot{3} \\ 7.\dot{6} \end{bmatrix}$$

4. a $\Pi = (p+s)Y - w(1+c)L$

$$= (p+s)f(L) - a(1+c)L^2$$

b. FONC : $\frac{d\Pi}{dL} = (p+s)f'(L) - 2a(1+c)L = 0$

ie $MR = MC$

c. $\frac{d^2\Pi}{dL^2} = (p+s)f''(L) - 2a(1+c)$

$\ominus \quad \oplus \quad \oplus$

$< 0 \quad \forall L \quad \therefore$ local & global max.

d. $\frac{dL^*}{ds}$? Method 1 : write FONC as

$$F = (p+s)f'(L(s)) - 2a(1+c)L(s) = 0$$

$$\frac{dFONC}{ds} = (p+s)f''(L)\frac{dL^*}{ds} + f'(L) - 2a(1+c)\frac{dL^*}{ds}$$

$$= 0 \Rightarrow \frac{dL^*}{ds} = \frac{-f'(L)}{\underbrace{(p+s)f''(L) - 2a(1+c)}_{<0 \text{ from SOSC}}} > 0$$

$\frac{dL^*}{dc}$? Method 2 (Implicit Fn. Rule)

$$\frac{dL^*}{dc} = \frac{-F_c}{F_L} = \frac{+2aL}{\underbrace{(p+s)f''(L) - 2a(1+c)}_{<0}} < 0$$

5. a. $dq^d = -b\,dp - b\,dt$

$dq^s = e\,dp$

$dq^d = dq^s + dm$

b. $\Rightarrow \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & -e \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} dq^d \\ dq^s \\ dp \end{bmatrix} = \begin{bmatrix} -b\,dt \\ 0 \\ dm \end{bmatrix}$

$A \quad \underset{\sim}{x} \quad = \quad d$

$|A| = 1(0-e) + b(0-1) = -(e+b)$

$$dp = \frac{\begin{vmatrix} 1 & 0 & -b\,dt \\ 0 & 1 & 0 \\ 1 & -1 & dm \end{vmatrix}}{|A|} = \frac{dm + b\,dt}{-(e+b)}$$

i) $\underline{dm > 0}$, $dt = 0$ $\Rightarrow dp = \frac{dm}{-(e+b)} \Rightarrow \frac{dp}{dm} = \frac{1}{-(e+b)} < 0$

ie $\uparrow m \Rightarrow \downarrow p$

ii $dm = 0$, $\underline{dt > 0}$ $\Rightarrow \frac{dp}{dt} = -\frac{b}{e+b} < 0$ ie $\uparrow t \Rightarrow p \downarrow$ (supply price)

c. $a - b(p+t) = c + ep + m$

$\Rightarrow -b\,dp - b\,dt = e\,dp + dm$ ✓

$dt = 0 \Rightarrow dp(e+b) = -dm \Rightarrow \frac{dp}{dp} = -\frac{1}{(e+b)}$

$dm = 0 \Rightarrow dp(e+b) = -b\,dt$

$\Rightarrow \frac{dp}{dt} = \frac{-b}{b+e}$ ✓

6. $A = \begin{bmatrix} a & 0 & 0 \\ 1 & b & 1 \\ 0 & 0 & c \end{bmatrix}$

Find all leading minors of A

$$|A_1| = a$$

$$|A_2| = \begin{vmatrix} a & 0 \\ 1 & b \end{vmatrix} = ab$$

$$|A_3| = |A| = abc$$

Positive definite if $|A_1|, |A_2|\ |A_3| > 0$

$|A_1| > 0 \Rightarrow a > 0$

$|A_2| > 0 \Rightarrow ab > 0 \Rightarrow$ b also > 0

$|A_3| > 0 \Rightarrow abc > 0 \Rightarrow c > 0$

Negative definite if $|A_1| < 0\ |A_2| > 0\ |A_3| < 0$

$|A_1| < 0 \Rightarrow a < 0$

$|A_2| > 0 \Rightarrow ab > 0 \Rightarrow b < 0$

$|A_3| < 0 \Rightarrow abc < 0 \Rightarrow c < 0$.