AUSTRALIAN NATIONAL UNIVERSITY

First Semester Examination - June 2004

MATHEMATICAL TECHNIQUES IN ECONOMICS 1
ECON 8013 / IDEC 8015/ ECON 4021

Study Period: 30 Minutes

Time Allowed: Three Hours

Permitted Materials: Nonprogrammable Calculators

Answer ALL questions. This exam is marked out of 140. Marks for individual questions are given at the end of each question.

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1. Consider the system of equations

$$x_1 - x_2 = -2$$
$$ax_1 + 2x_2 = 4$$

where *a* is some real number. Use matrix notation and the concept of rank to find the value of *a* that yields *no* unique solution for this system of equations. In this case will there be an infinite number of solutions or will they be inconsistent. Illustrate your answer.

[10 marks]

2. A monopolist's demand is given by $p(x) = 80 - x$ while costs are given by $c(x) = x^2$. The monopolist is subject to a tax rate of *t* so profits can be expressed as

$$\Pi = [p(x) - t]x - c(x)$$

a. Write down the first-order condition for the firm's profit-maximisation problem and solve for the optimal level of output, *x*. Check that the second-order condition is satisfied.

b. The monopolist is lobbying the government for a tax reduction, arguing that output expansion and price reduction will follow. Show how *x* and *p* will change if *t* falls. Can you support his argument?

[13 marks]

3. Determine whether the functions $y = x_1 x_2^2$ and $y = x_1^{1/4} x_2^{1/2}$, $x_1, x_2 > 0$, are strictly concave and/or strictly quasiconcave.

[17 marks]

4. Consider the following closed-economy macroeconomic model in which the *IS*, *LM* and "supply-side" equations are, respectively:

$$Y = C(Y, A) + I(r), \quad 0 < C_y < 1, \; C_A > 0, \quad I'(\quad) < 0$$

$$\frac{M}{P} = L(Y, r), \quad L_Y > 0, \quad L_r < 0$$

$$Y = f(W/P), \; f'(W/P) < 0$$

where *Y* is real output, *A* denotes the assets of households, *C* is consumption, *I* is investment, *r* is the interest rate, M is the nominal money supply, *P* is the price level and *W* is the nominal wage rate. In this model, *A*, *M* and *W* are exogenous and *Y*, *r* and *P* are endogenous. Assume throughout the analysis that *W* does not change.

a. Take total differentials of the three equations above and express in matrix form.

b. A stock market crash leads to a fall in *A*. Find the comparative static effects of this crash on the level of *Y*.

c. Now suppose that the government uses monetary policy to offset the fall in *Y* caused by the stock market crash (so that $dY = 0$). Show that dM is unambiguously positive. Then find how interest rates change in this case. Can you say whether they rise or fall? Illustrate your answer.

[25 marks]

5. Consider a firm with the production function $Y = K^{1/2} + L^{1/2}, \quad K > 0, \; L > 0$.

a. Determine whether the function is homogeneous and, if so, state its degree of homogeneity.

b. Find the marginal rate of technical substitution, $MRTS_{LK} = \left.\frac{dK}{dL}\right|_{\bar{y}} = -\frac{f_L}{f_K}$.

Use this to determine whether the function is homothetic. Find the curvature of the isoquant $d^2K/dL^2\big|_{\bar{Y}}$. Sketch two isoquants for this production function to illustrate your findings.

c. Find $Kf_K + Lf_L$ and relate your answer to Euler's Theorem.

[18 marks]

6. A consumer has the utility function

$$U(x_1, x_2) = (x_1 - c_1)^{1/2}(x_2 - c_2)^{1/2}$$

where c_i is the minimum amount of good *i* required for subsistence. Using $\tilde{x}_i = x_i - c_i$ we can express the utility maximisation problem as:

$$\max \; \tilde{x}_1^{1/2}\tilde{x}_2^{1/2} \; \text{ s.t } \; p_1(\tilde{x}_1 + c_1) + p_2(\tilde{x}_2 + c_2) = m$$

a. Use the Lagrange method to show that the individual's Marshallian demand functions are: $x_1^* = \tilde{x}_1^* + c_1 = \frac{m - p_2 c_2}{2p_1} + \frac{c_1}{2}$, $x_2^* = \tilde{x}_2^* + c_2 = \frac{m - p_1 c_1}{2p_2} + \frac{c_2}{2}$.

b. Check that the second-order sufficient conditions for a local maximum are satisfied (noting that if they are satisfied for $\tilde{x}_i^*$ then they are satisfied for x_i^*).

c. Show that the indirect utility function is

$$V(p_1, p_2, m) = \frac{1}{2} p_1^{-1/2} p_2^{-1/2} (m - p_1 c_1 - p_2 c_2)$$

d. In order to obtain the level of utility u_0 what level of income would be required? What is the expenditure function?

e. Use Shepherd's Lemma to find the Hicksian compensated demand curve for x_2. Explain how this relates to the Envelope Theorem. Demonstrate that Hicksian demand curves always slope down.

[27 marks]

7. A consumer has the utility function

$$u(x_1, x_2) = x_1 x_2$$

where x_1 is apples and x_2 is cheese. The price of an apple is \$2 while the price of a piece of cheese is \$3. She has an income of \$45 (per day) such that her *budget constraint* is given by

$$2x_1 + 3x_2 \le 45 .$$

In addition to her budget constraint, the consumer is on a strict diet: she must consume no more than 1200 calories each day. Each apple yields 50 calories while each piece of cheese yields 100 calories. Therefore her *calorie constraint* is given by:

$$50x_1 + 100x_2 \le 1200 .$$

a. Write down the Lagrangian and the corresponding Kuhn-Tucker conditions. Assume that the second-order conditions are satisfied. [See next page.]

b. Sketch the two constraints and identify and illustrate the five possible optimal solutions.

c. Verify (by working through all the possibilities) that the case in which $x_1 > 0, x_2 > 0$, the calorie constraint binds but the budget constraint does not yields the optimal solution.

[30 marks]

The Kuhn-Tucker Theorem

Given the problem:

$$\max\ f(x_1, x_2, ..., x_n) \text{ s.t. } \begin{aligned} &g^1(x_1, x_2, ..., x_n) \geq 0, \\ &g^2(x_1, x_2, ..., x_n) \geq 0 \\ &\vdots \\ &g^m(x_1, x_2, ..., x_n) \geq 0 \\ &x_1, x_2 \geq 0 \end{aligned}$$

if all the functions *f* and g^j, $j = 1,\ldots,m$ are concave and differentiable, and if there exists a point $(x_1^0, x_2^0, ..., x_n^0)$ such that $g^j(x_1^0, x_2^0, ..., x_n^0) > 0$ for all $j = 1,\ldots,m$, then there exist *m* Lagrange multipliers λ_j^* such that the following conditions are necessary and sufficient for the point $(x_1^*, x_2^*, ..., x_n^*)$ to be a solution for this problem:

$$f_i(x_1^*, x_2^*, ..., x_n^*) + \sum \lambda_j^* g_i^j(x_1^*, x_2^*, ..., x_n^*) \leq 0,\ x_i^* \geq 0$$

$$x_i^*(f_i + \sum \lambda_j^* g_i^j) = 0,\ \ i = 1, ..., n$$

$$g^j(x_1^*, x_2^*, ..., x_n^*) \geq 0,\ \lambda_j^* \geq 0$$

$$\lambda_j^* g^j(x_1^*, x_2^*, ..., x_n^*) = 0,\ \ j = 1, ..., m$$
