AUSTRALIAN NATIONAL UNIVERSITY

First Semester Examination - June 2003

MATHEMATICAL TECHNIQUES IN ECONOMICS 1
ECON 8013 / ECDV 8113/ ECON 4021

Study Period: 30 Minutes

Time Allowed: Three Hours

Permitted Materials: Nonprogrammable Calculators

Answer ALL questions. This exam is marked out of 120. Marks for individual questions are given at the end of each question.

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1. The total cost to a firm of producing a good, *x*, is given by $C(x) = 2x^2 + 8$. The average cost is given by $AC(x) = C(x)/x,\ x > 0$. Find $MC(x),\ AC(x),\ dAC/dx$ and d^2AC/dx^2. Use your answers to comment on the concavity/convexity of the average cost function. Sketch graphs of the marginal cost and average cost functions.

[10 marks]

2. Consider the following CES utility function

$$U = [\delta x^{-r} + (1-\delta)y^{-r}]^{-1/r},\quad r > -1,\ 0 < \delta < 1$$

a. What is the marginal utility of *x*? What is the marginal utility of *y*?

b. Find the marginal rate of substitution between goods *x* and *y*.

[8 marks]

3. Suppose that the IS curve for an economy is given by the following expression:

$$Y = C(Y) + I(r) + G,\quad 0 < C'(Y) < 1,\ I'(r) < 0$$

where *Y* is national income, *r* is the interest rate, *C* is aggregate consumption, *I* is investment and *G* is government spending.

a. Take total differentials to find linear approximations to the horizontal and vertical shifts of the IS curve in response to an increase in G. Illustrate your results diagrammatically.

b. Suppose that the LM curve is as follows:

$$\frac{M}{P} = L(Y, r),\quad L_Y > 0,\ L_r < 0$$

where *M* is the nominal money supply and *P* is the price level. Take total differentials of the LM curve, assuming that *M* and *P* can both vary.

c. Assume that in this economy Y and r are endogenous and G, P and M are exogenous. If M increases by dM and G increases by dG, find dY and dr for the economy as a whole. Can you say whether Y rises or falls? What about r? Illustrate your results diagrammatically.

[20 marks]

4. An American-based Finnish firm sells its output in a competitive market at the exogenously given price, p. Its production function is

$$X = f(L_1 + L_2), \quad f'(\,) > 0, \;\; f''(\,) < 0$$

where X is the amount of output produced, L_1 is the amount of Finnish-speaking labour employed and L_2 is the amount of American labour employed. The firm is a monopsonist in the labour market for Finnish speakers, with the following supply curve:

$$w_1 = a + bL_1, \quad a, b > 0$$

where w_1 is the wage rate paid to Finnish-speakers. The American labour market is highly competitive, so the firm must take its wage as given at w_2.

a. State the firm's profits as a function of L_1 and L_2.

b. Find the first-order conditions which determine the profit-maximising choices of L_1 and L_2, L_1* and L_2*. At L_1* and L_2* what is the marginal cost of hiring each type of labour?

c. Show that the second-order conditions for a local interior maximum for profits are satisfied. Is it also a unique, global maximum?

d. If w_2 rises, will L_1* rise, fall or remain the same? What about L_2*?

[20 marks]

5. A consumer has the following utility function:

$$U(x_1, x_2) = x_1 e^{x_2}$$

a. Suppose the consumer maximises utility subject to the constraint that $I = p_1 x_1 + p_2 x_2$. Use the Lagrange method to write down the consumer's optimisation problem and first-order conditions. Show that the individual's Marshallian demand functions are $x_1^* = \frac{p_2}{p_1}$ and $x_2^* = \frac{I}{p_2} - 1$ (assume $\frac{I}{p_2} > 1$).

b. Check that the second-order sufficient conditions for a local maximum are satisfied.

c. Check whether the Marshallian demand functions for x_1 and x_2 are homogeneous with respect to the three variables (p_1, p_2, I). Are the functions homothetic?

d. Find the level of (indirect) utility, V, when quantities are chosen optimally as a function of (p_1, p_2, I) and show how it relates to the value of the Lagrange multiplier. Confirm that the Marshallian demand for x_1 can be found using Roy's Identity.

e. In order to obtain the level of utility u_0 what level of income would be required? What is the expenditure function?

f. Use Shepherd's Lemma to find the Hicksian compensated demand curves for x_1 and x_2. Show that $\partial x_1^c(p_1, p_2, u_0)/\partial p_2$ is always positive. Illustrate this for the case where p_2 falls.

[30 marks]

6. Suppose a firm's production function is given by

$$Q = L^{1/2} + K^{1/2}$$

a. Find the slope of an isoquant $\left.\frac{dK}{dL}\right|_{dY=0}$ and the curvature of an isoquant $\left.\frac{d^2K}{dL^2}\right|_{dY=0}$. Are the isoquants either strictly concave or strictly convex? Sketch an isoquant and use your diagram to explain whether the production function is strictly quasiconcave or strictly quasiconvex.

b. Confirm your answer to *a.* using a bordered Hessian matrix, recalling that a function is strictly quasiconcave if $|\bar{\mathbf{H}}| > 0$ and strictly quasiconvex if $|\bar{\mathbf{H}}| < 0$.

[15 marks]

7. Suppose that a monopolist can supply its market from two plants, with cost functions

$$C_1 = 5x_1, \quad C_2 = 6x_2$$

Suppose also that the monopolist faces linear demand

$$p = 200 - (x_1 + x_2)$$

Therefore the firm's profit-maximising problem is to

$$\max \Pi(x_1, x_2) = [200 - (x_1 + x_2)](x_1 + x_2) - 5x_1 - 6x_2 \text{ s.t. } 0 \le x_1, 0 \le x_2$$

Note that there are 3 solution possibilities (excluding the case where both outputs are zero): (i) $x_1 > 0, x_2 > 0$, (ii) $x_1 > 0, x_2 = 0$, (iii) $x_1 = 0, x_2 > 0$.

a. Express clearly the conditions that must be satisfied by the profit maximising outputs, x_1 and x_2.. Show that only case (ii) above is possible. How much output will be produced in Plant 1?

b. Now consider the case where both plants have a maximum capacity of 60 units of output. What is the firm's profit-maximising problem now? Express clearly the conditions that must be satisfied by the profit-maximising outputs, x_1 and x_2. Assuming that Plant 1 will produce to capacity, what level of output will be produced in Plant 2?

Hint: It may be useful to recall the following Theorem:

Theorem 3

If $\mathbf{x}^*$ is a solution to the problem $\max_{x_i} y = f(\mathbf{x})$ s.t. $a_i \le x_i \le b_i,\ i = 1, \ldots, n$ then it satisfies one or both of

$$f_i(\mathbf{x}^*) \le 0 \text{ and } (x_i^* - a_i) f_i(\mathbf{x}^*) = 0$$
$$f_i(\mathbf{x}^*) \ge 0 \text{ and } (b_i - x_i^*) f_i(\mathbf{x}^*) = 0$$

for all $i = 1, \ldots, n$. If $a_i < x_i^* < b_i$ then *both* conditions hold.

[17 marks]
