Mid-Semester Exam, 2005

1. $y = \ln x$

A function is strictly concave if $\forall x', x''$ & $\forall \lambda \in (0,1)$

$$f(\bar{x}) > \lambda f(x') + (1-\lambda) f(x'')$$

where $\bar{x} = \lambda x' + (1-\lambda) x''$

Here $x' = 1 \quad x'' = 2$

$$\Rightarrow \bar{x} = \tfrac{1}{4} \cdot 1 + \tfrac{3}{4} \cdot 2 = \tfrac{7}{4} = 1.75$$

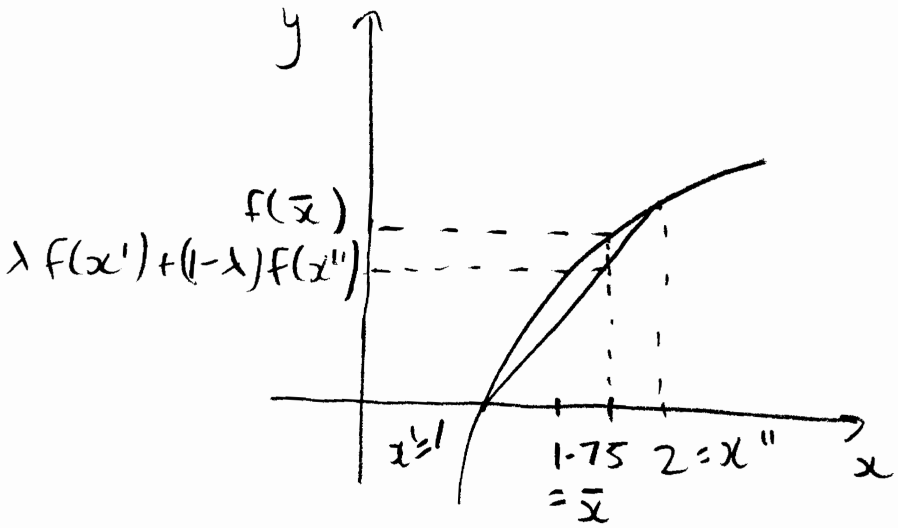
$$\therefore f(\bar{x}) = \ln \tfrac{7}{4} = 0.56$$

$$f(x') = f(1) = 0$$

$$f(x'') = f(2) = \ln 2 = 0.693$$

$$\therefore \tfrac{1}{4} \cdot 0 + \tfrac{3}{4} \cdot 0.693 = 0.520 < f(\bar{x})$$

$\therefore$ $y = \ln x$ is strictly concave.



2. $f(x) = \frac{1}{x-1}$

a. $\lim_{x \to 1^-} f(x) = -\infty$ $\qquad \lim_{x \to \infty} f(x) = 0$

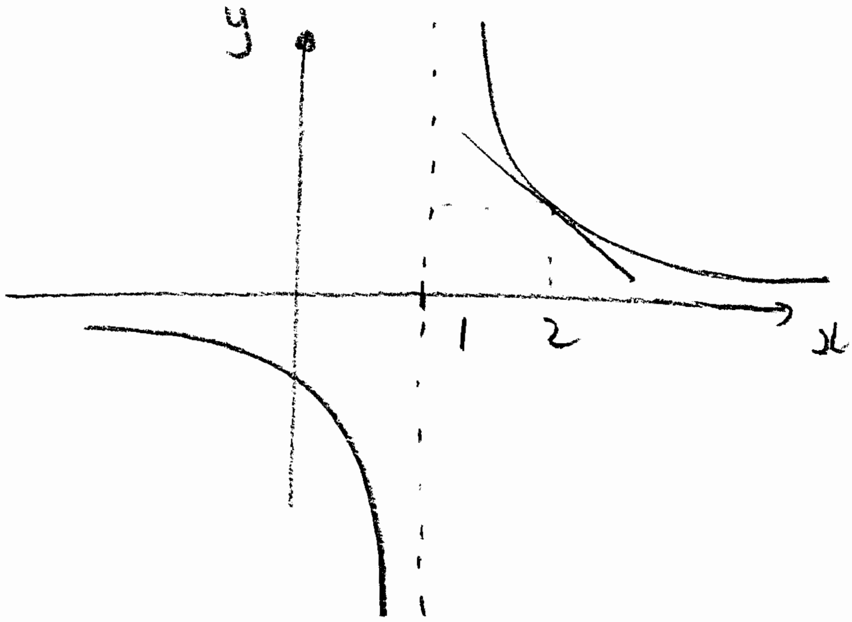
$\lim_{x \to 1^+} f(x) = \infty$ $\qquad \lim_{x \to -\infty} f(x) = 0$

$\Rightarrow \therefore$ No limit $\therefore$ Not continuous.

b. $f'(x) = -(x-1)^{-2} < 0 \quad \forall x$

$f''(x) = 2(x-1)^{-3} > 0 \quad \forall x > 1 \Rightarrow$ s convex

$< 0 \quad \forall x < 1 \Rightarrow$ s. concave



3. $$p = q^{-1/\varepsilon}$$

$$TR = pq = q^{1-1/\varepsilon}$$

$$TC = c.q$$

a. $$\pi = TR - TC$$

$$= q^{1-1/\varepsilon} - c.q$$

<u>FOC</u>. $$\frac{d\pi}{dq} = \left(1-\frac{1}{\varepsilon}\right)q^{-1/\varepsilon} - c = 0$$

$$\Rightarrow q^* = \left(\frac{c}{1-1/\varepsilon}\right)^{-\varepsilon}$$

<u>SOSC</u> $$\frac{d^2\pi}{dq^2} = \underbrace{-\frac{1}{\varepsilon}}_{-}\left(1-\frac{1}{\varepsilon}\right)\underbrace{q^{-1/\varepsilon-1}}_{+}$$

needs to be positive for $\frac{d^2\pi}{dq^2} < 0$

$$1-\frac{1}{\varepsilon} > 0 \Rightarrow \frac{1}{\varepsilon} < 1 \Rightarrow \varepsilon > 1$$

$\therefore$ we need $\varepsilon > 1$ for SOSC to be satisfied.

b. $$p^* = (q^*)^{-1/\varepsilon} = \frac{c}{1-1/\varepsilon} = c\left(1-\underbrace{1/\varepsilon}_{\varepsilon^{-1}}\right)^{-1}$$

$$\therefore \frac{dp^*}{d\varepsilon} = \underset{+}{-c}\,\underset{+}{\left(1-1/\varepsilon\right)^{-2}} \times \underset{+}{\varepsilon^{-2}}$$

< 0.

i.e The more inelastic the demand, the higher the mark-up

4.

a. (1)' $dL_A + \gamma dW = \beta_A dY$

(2)' $dL_B + \gamma dW = \beta_B dY$

3' $dL_A + dL_B = d\bar{L}$

b.
$$\begin{bmatrix} 1 & 0 & \gamma \\ 0 & 1 & \gamma \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} dL_A \\ dL_B \\ dW \end{bmatrix} = \begin{bmatrix} \beta_A dY \\ \beta_B dY \\ d\bar{L} \end{bmatrix}$$

$dY > 0 \quad d\bar{L} = 0$

$$\Rightarrow dL_A = \frac{\begin{vmatrix} \beta_A dY & 0 & \gamma \\ \beta_B dY & 1 & \gamma \\ 0 & 1 & 0 \end{vmatrix}}{|A|} = \frac{\beta_A dY(-\gamma) + \gamma(\beta_B dY)}{-2\gamma}$$

$$= dY \frac{(\beta_A - \beta_B)}{2}$$

$$\therefore \frac{dL_A}{dY} = \frac{\beta_A - \beta_B}{2} > 0 \text{ when } \beta_A > \beta_B$$

By symmetry $\frac{dL_B}{dY} = \frac{\beta_B - \beta_A}{2} < 0$ when $\beta_A > \beta_B$

(Interpret: When labor demand in Firm A is <u>more</u> responsive to overall demand, an increase in demand causes labor dd to rise in A (job creation) & fall in B (Job destruction))

c. $d\bar{L} > 0 \quad dY = 0$

$$dw = \frac{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & d\bar{L} \end{vmatrix}}{|A|} = \frac{d\bar{L}}{-2\gamma}$$

$$\therefore \frac{dw}{d\bar{L}} = -\frac{1}{2\gamma}$$

So when $\bar{L}\uparrow$, $w\downarrow$ by $\frac{1}{2\gamma}$.

d. Sub (1) & (2) into (3) $\Rightarrow$

$$\bar{L} = -\gamma W + \beta_A Y - \gamma W + \beta_B Y$$

$$d\bar{L} = -2\gamma\, dW + \beta_A\, dY + \beta_B\, dY$$

$dY = 0 \Rightarrow \quad \frac{dW}{d\bar{L}} = -\frac{1}{2\gamma}$

5.

Answer :

$a \neq 0 \quad a \neq 7 \quad$ unique

$\left.\begin{array}{ll} a = 0 & b = 9/2 \\ a = 7 & b = 19/3 \end{array}\right\} \Rightarrow \infty$ sdns.

otherwise $\rightarrow$ no solutions

method :

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ -x_1 + a x_2 - 21 x_3 &= 2 \\ 3x_1 + 7x_2 + a x_3 &= b \end{aligned}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & a & -21 \\ 3 & 7 & a \end{bmatrix}$$

$$\begin{aligned} |A| &= 1(a^2 + 21 \times 7) - 2(-a + 21 \times 3) + 3(-7 - 3a) \\ &= a^2 + 147 + 2a - 126 - 21 - 9a \\ &= a^2 - 7a \quad = 0 \quad \text{if } a = 0, 7 \end{aligned}$$

$\therefore$ unique solutions if $|A| \neq 0$

$\Rightarrow a \neq 0, 7.$

$$B = \begin{bmatrix} 1 & 2 & 3 & 1 \\ -1 & a & -21 & 2 \\ 3 & 7 & a & b \end{bmatrix}$$

Look at $\begin{vmatrix} 2 & 3 & 1 \\ a & -21 & 2 \\ 7 & a & b \end{vmatrix} = \begin{array}{l} 2(-21b-2a) \\ -3(ab-14)+a^2+147 \end{array}$

$= -42b - 4a - 3ab$
$+ 42 + a^2 + 147$

For $a=0$: $-42b + 189 = 0$ when
$b = 9/2$ or 4.5

ie If $a=0$ & $b=4.5$ Rank A = Rank B
$\Rightarrow \infty$ solutions

If $a=7$: $-42b - 28 - 21b + 42 + 49$
$+ 147$
$= -63b + 210$
$= 0$ if $b = 3\frac{1}{3}$

so if $a=7$, $b = 10/3$ Rank A = Rank B
$\Rightarrow \infty$ solutions

For all other $a \neq 0, 7$ & $b \neq 4.5, 10/3$,
Rank B = 3 $\Rightarrow$ 0 solutions
(equations are inconsistent)

6. Use implicit differentiation to find $\frac{dy}{dx}$ for the function $e^{x^2+y} - 5 = 0$.

Take the total differential & use the Implicit Function Rule to confirm your answer.

Implicit differentiation:

$$\left(2x + \frac{dy}{dx}\right) e^{x^2+y} = 0$$

e^{x^2+y} cannot $= 0 \Rightarrow 2x + \frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = -2x$$

TD: $F(x,y) = e^{x^2+y} - 5 = 0$

$$dF = F_x dx + F_y dy$$

$$= 2x \cancel{e^{x^2+y}} dx + \cancel{e^{x^2+y}} dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -2x$$

1. Suppose that the amount of capital at time t, $K = K(t)$ and the efficiency with which it is used affect total output according to the function

$$Y = f(K, t) = 0.2(1+t)^{1/2} K$$

$$\text{where } K = K_0 e^{0.05t}$$

Find & give the economic intuition of the total and partial derivatives of Y with respect to t.

Partial $$\frac{\partial Y}{\partial t} = 0.1(1+t)^{-1/2} . K \quad > 0.$$

i.e. Holding K constant, output grows at this rate.

Total $$\frac{dY}{dt} = \frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial K} . \frac{\partial K}{\partial t}$$

$$= (0.1)(1+t)^{-1/2} . K + (0.2)(1+t)^{1/2} . 0.05 K_0 e^{0.05t}$$

$$= (0.1)(1+t)^{-1/2} K + \underbrace{(0.01)(1+t)^{1/2} . K}_{>0}$$

Allowing for the capital stock to change over time (ie to grow exponentially)), the total derivative is larger than the partial derivative: output growth is higher because of the growth in the K stock.

Or: $\partial Y/\partial t$: how time passing affects output. $\frac{\partial Y}{\partial K} . \frac{\partial K}{\partial t}$: how ↑K stock over time impacts on output.

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